

# CMB Bispectrum from Primordial Scalar, Vector and Tensor Non-Gaussianities

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We present an all-sky formalism for the Cosmic Microwave Background (CMB) bispectrum induced by the primordial non-Gaussianities not only in scalar but also in vector and tensor fluctuations. We find that the bispectrum can be formed in an explicitly rationally invariant way by taking into account the angular and polarization dependences of the vector and tensor modes. To demonstrate this and present how to use our formalism, we consider a specific example of the correlation between two scalars and a graviton as the source of non-Gaussianity. As a result, we show that the CMB reduced bispectrum of the intensity anisotropies is evaluated as a function of the multipole and the coupling constant between two scalars and a graviton denoted by  $g_{tss}$ ;  $|b_{\ell\ell\ell}| \sim \ell^{-4} \times 8 \times 10^{-18} |g_{tss}|$ . By estimating the signal-to-noise ratio, we find that the constraint as  $|g_{tss}| < 6$  will be expected from the PLANCK experiment.

## §1. Introduction

Bispectrum (three-point correlation functions) of the Cosmic Microwave Background (CMB) temperature anisotropies has been attracting attention over the years as a powerful observational tool to investigate the primordial non-Gaussianity.<sup>1),2)</sup> As is well known, primordial curvature perturbations whose statistics deviate from the pure Gaussian ones can produce the nonzero bispectrum of the CMB temperature anisotropies. The size of the primordial non-Gaussianity has often been parametrized by a so-called nonlinearity parameter  $f_{\text{NL}}$ . Depending on the shape of the bispectrum, this nonlinearity parameter can be sorted into three types of  $f_{\text{NL}}$ , called local, equilateral and orthogonal types. Current observational limits on these  $f_{\text{NL}}$ s are given by  $-10 < f_{\text{NL}}^{\text{local}} < 74$  for the local type,  $-214 < f_{\text{NL}}^{\text{equil}} < 266$  for the equilateral type and  $-410 < f_{\text{NL}}^{\text{orthog}} < 6$  for the orthogonal type (95 % C.L.).<sup>3)</sup> These observational constraints are still consistent with the Gaussian primordial curvature perturbations that are expected to be generated from the standard single slow-roll inflation model. However, one can also consider that there may be some cue for hunting the non-Gaussianity because the central values of some  $f_{\text{NL}}$ s have deviated from zero. Hence, if future experiments would confirm that the statistics of the primordial curvature perturbations deviate from the Gaussian ones, then the standard single slow-roll inflation model can be excluded as a dominant mechanism of generating primordial fluctuations. Thus, the primordial non-Gaussianity can be considered as a new probe of the mechanism of generating primordial curvature perturbations.

As discussed above, previous studies have focused largely on non-Gaussianities in scalar-mode perturbations. However, we can also consider the sources of non-Gaussianities in vector and tensor perturbations, for example, the nonlinear cou-

plings between gravitons and scalars during inflation,<sup>4)</sup> nonlinearities of the Sachs-Wolfe effect,<sup>5),6),7),8)</sup> cosmic strings,<sup>9),10)</sup> and primordial magnetic fields.<sup>11),12),13),14),15),16),17)</sup> Hence, in order to constrain the size of non-Gaussianities and understand the nature of these sources, one should include the contribution of vector and tensor non-Gaussianities to the CMB bispectrum.

In our previous work,<sup>18)</sup> we presented the bispectrum formulae of the CMB temperature and polarization anisotropies sourced from non-Gaussianity not only in scalar but also in vector and tensor fluctuations. It is expected that the nonlinearities will also induce the tensor and vector mode perturbations, and the modes may generate more characteristic features in the CMB angular spectra than in the scalar one. In Ref. 18) , we found that the bispectrum formulae for vector and tensor modes in all sky analysis formally take complicated forms compared with the scalar mode case owing to the dependence of the photon transfer functions on the azimuthal angle between the wave vector of photon fluctuation  $\mathbf{k}$  and the unit vector specifying the line of sight direction  $\hat{\mathbf{n}}$ . However, by using flat sky approximation, we have simplified the equations of bispectra of the CMB anisotropies to solve the above difficulty because no azimuthal dependence arises in this limit.

This paper is an extension of our previous work. We present a general formalism of the CMB bispectrum induced from the primordial vector and tensor fluctuations in the all-sky analysis. We newly consider the angular dependences in the polarization vector and tensor bases, which have been neglected in our previous work. To demonstrate how to calculate the CMB bispectrum by making use of our formula, we show a calculation of the CMB bispectrum induced from the primordial non-Gaussianity generated through the interaction between two scalars and a graviton (tensor) during inflation, which has been originally discussed by Maldacena.<sup>4)</sup>

This paper is organized as follows. In the next section, we present a formulation of the CMB bispectrum in the all-sky approach, which is an extension of our previous work.<sup>18)</sup> In §3, we show the calculation of the CMB bispectrum induced from the primordial non-Gaussianity generated through the two scalars and a graviton (tensor) correlator during inflation. We define a coupling parameter characterizing the strength of such interaction as  $g_{tss}$  and evaluate an observational limit on  $g_{tss}$  by calculating the signal-to-noise ratio. In the final section, we give a summary and conclusion of this paper.

## **§2. Formulation of the CMB bispectra for scalar, vector and tensor modes**

In this section, we derive general formulae of the CMB bispectrum of temperature and polarization fluctuations induced by the primordial non-Gaussianity in scalar, vector or tensor-mode perturbations in the all-sky analysis.

At first, we introduce an expression of CMB fluctuation. In the all-sky analysis, CMB fluctuations of intensity or polarization field are expanded with the spin-0 or spin-2 spherical harmonics, respectively.<sup>18),19),20)</sup> Then, the coefficients of CMB

fluctuations, called  $a_{\ell m}$ , are described as

$$a_{X,\ell m}^{(Z)} = 4\pi(-i)^\ell \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda [\text{sgn}(\lambda)]^{\lambda+x} {}_{-\lambda}Y_{\ell m}^*(\hat{\mathbf{k}}) \xi^{(\lambda)}(\mathbf{k}) \mathcal{T}_{X,\ell}^{(Z)}(k), \quad (2.1)$$

where the index  $Z$  denotes the mode of perturbations:  $Z = S$  (scalar),  $= V$  (vector) or  $= T$  (tensor) and its helicity is expressed by  $\lambda$ ;  $\lambda = 0$  for  $(Z = S)$ ,  $= \pm 1$  for  $(Z = V)$  or  $= \pm 2$  for  $(Z = T)$ ,  $X$  discriminates between intensity and two polarization (electric and magnetic) modes, respectively, as  $X = I, E, B$  and  $x$  is determined by it:  $x = 0$  for  $X = I, E$  or  $= 1$  for  $X = B$ ,  $\xi^{(\lambda)}$  is the initial perturbation decomposed on each helicity state and  $\mathcal{T}_{X,\ell}^{(Z)}$  is the time-integrated transfer function in each sector (calculated in, for example, Refs. 18), 21), 22)).\*)

Next, we expand  $\xi^{(\lambda)}$  with spin- $(-\lambda)$  spherical harmonics as

$$\xi^{(\lambda)}(\mathbf{k}) \equiv \sum_{\ell m} \xi_{\ell m}^{(\lambda)}(k) {}_{-\lambda}Y_{\ell m}(\hat{\mathbf{k}}), \quad (2.2)$$

and eliminate the angular dependence in Eq. (2.1) by performing  $\hat{\mathbf{k}}$ -integral:

$$a_{X,\ell m}^{(Z)} = 4\pi(-i)^\ell \int_0^\infty \frac{k^2 dk}{(2\pi)^3} \sum_\lambda [\text{sgn}(\lambda)]^{\lambda+x} \xi_{\ell m}^{(\lambda)}(k) \mathcal{T}_{X,\ell}^{(Z)}(k). \quad (2.3)$$

Here, we use the orthogonality relation of spin- $\lambda$  spherical harmonics as<sup>23), 24)</sup>

$$\int d^2\hat{\mathbf{n}} {}_\lambda Y_{\ell' m'}^*(\hat{\mathbf{n}}) {}_\lambda Y_{\ell m}(\hat{\mathbf{n}}) = \delta_{\ell, \ell'} \delta_{m, m'}. \quad (2.4)$$

The initial bispectrum in vector or tensor-mode perturbations will be expressed as

$$\left\langle \prod_{i=1}^3 \xi^{(\lambda_i)}(\mathbf{k}_i) \right\rangle \equiv (2\pi)^3 F^{\lambda_1 \lambda_2 \lambda_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \delta\left(\sum_{i=1}^3 \mathbf{k}_i\right). \quad (2.5)$$

This definition, which includes the angular dependence on  $\mathbf{k}$  in the polarization vector or tensor, is more general than Eq. (3) of Ref. 18). We will see in the discussion in §3 that the initial bispectrum from inflation indeed takes the above form for the tensor case. On the other hand, to calculate the CMB bispectrum using Eq. (2.3), the bispectrum of  $\xi_{\ell m}$  is needed. If the primordial bispectrum satisfies the rotational invariance, we can set it as

$$\left\langle \prod_{i=1}^3 \xi_{\ell_i m_i}^{(\lambda_i)}(k_i) \right\rangle \equiv (2\pi)^3 \mathcal{F}_{\ell_1 \ell_2 \ell_3}^{\lambda_1 \lambda_2 \lambda_3}(k_1, k_2, k_3) \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \quad (2.6)$$

Then, using Eq. (2.2), the conversion equation between  $F^{\lambda_1 \lambda_2 \lambda_3}$  and  $\mathcal{F}_{\ell_1 \ell_2 \ell_3}^{\lambda_1 \lambda_2 \lambda_3}$  is derived as

$$\mathcal{F}_{\ell_1 \ell_2 \ell_3}^{\lambda_1 \lambda_2 \lambda_3}(k_1, k_2, k_3) = \sum_{m_1 m_2 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \left[ \prod_{i=1}^3 \int d^2\hat{\mathbf{k}}_i {}_{-\lambda_i}Y_{\ell_i m_i}^*(\hat{\mathbf{k}}_i) \right]$$

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\*) Here, we set  $0^0 = 1$ .

$$\times F^{\lambda_1 \lambda_2 \lambda_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \delta \left( \sum_{i=1}^3 \mathbf{k}_i \right). \quad (2.7)$$

From Eqs. (2.3), (2.6) and the orthogonality of Wigner-3j symbols as Eq. (A.5), the CMB angle-averaged bispectrum, which is defined as<sup>1),2)</sup>

$$B_{X_1 X_2 X_3, \ell_1, \ell_2, \ell_3}^{(Z_1 Z_2 Z_3)} \equiv \sum_{m_1 m_2 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \left\langle \prod_{i=1}^3 a_{X_i, \ell_i m_i}^{(Z_i)} \right\rangle, \quad (2.8)$$

can be written as

$$B_{X_1 X_2 X_3, \ell_1, \ell_2, \ell_3}^{(Z_1 Z_2 Z_3)} = \left[ \prod_{n=1}^3 4\pi(-i)^{\ell_n} \int_0^\infty \frac{k_n^2 dk_n}{(2\pi)^3} \mathcal{T}_{X_n, \ell_n}^{(Z_n)}(k_n) \sum_{\lambda_n} [\text{sgn}(\lambda_n)]^{\lambda_n + x_n} \right] \times (2\pi)^3 \mathcal{F}_{\ell_1 \ell_2 \ell_3}^{\lambda_1 \lambda_2 \lambda_3}(k_1, k_2, k_3). \quad (2.9)$$

Thus, when one computes the CMB bispectrum, only the alternative initial bispectrum  $\mathcal{F}_{\ell_1 \ell_2 \ell_3}^{\lambda_1 \lambda_2 \lambda_3}$  is necessary in each case.

### §3. CMB bispectrum induced by the primordial non-Gaussianity in the two scalars and a graviton correlator

In this section, we demonstrate how to calculate  $\mathcal{F}_{\ell_1 \ell_2 \ell_3}^{\lambda_1 \lambda_2 \lambda_3}$  and the CMB bispectrum by considering the contribution of two scalars and a graviton correlator.<sup>4)</sup> Furthermore, we evaluate an observational limit on the primordial non-Gaussianity of the graviton sector by calculating the signal-to-noise ratio.

#### 3.1. Two scalars and a graviton interaction during inflation

We consider a general single-field inflation model with Einstein-Hilbert action<sup>25)</sup>:

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{pl}}^2}{2} R + p(\phi, X) \right], \quad (3.1)$$

where  $g$  is the determinant of the metric,  $R$  is the Ricci scalar,  $M_{\text{pl}}^2 \equiv 1/(8\pi G)$ ,  $\phi$  is a scalar field, and  $X \equiv -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2$ . Using the background equations, the slow-roll parameter and the sound speed for perturbations are given by

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{X p_{,X}}{H^2 M_{\text{pl}}^2}, \quad c_s^2 \equiv \frac{p_{,X}}{2X p_{,XX} + p_{,X}}, \quad (3.2)$$

where  $H$  is the Hubble parameter, the dot means a derivative with respect to the physical time  $t$  and  $p_{,X}$  denotes partial derivative of  $p$  with respect to  $X$ . We write a metric by ADM formalism

$$ds^2 = -N^2 dt^2 + a^2 e^{\gamma_{ab}} (dx^a + N^a dt)(dx^b + N^b dt), \quad (3.3)$$

where  $N$  and  $N^a$  are respectively the lapse function and shift vector,  $\gamma_{ab}$  is a transverse and traceless tensor as  $\gamma_{aa} = \partial_a \gamma_{ab} = 0$ , and  $e^{\gamma_{ab}} \equiv \delta_{ab} + \gamma_{ab} + \gamma_{ac} \gamma_{cb} / 2 + \dots$ .

On the flat hypersurface, the gauge-invariant curvature perturbation  $\zeta$  is related to the first-order fluctuation of the scalar field  $\varphi$  as  $\zeta = -H\varphi/\dot{\phi}$ . Following the conversion equations (B·20) and (B·27), we decompose  $\zeta$  and  $\gamma_{ab}$  into the helicity states as

$$\xi^{(0)}(\mathbf{k}) = \zeta(\mathbf{k}) , \quad \xi^{(\pm 2)}(\mathbf{k}) = \frac{1}{2}e_{ab}^{(\mp 2)}(\hat{\mathbf{k}})\gamma_{ab}(\mathbf{k}) . \quad (3.4)$$

Here,  $e_{ab}^{(\pm 2)}$  is a transverse and traceless polarization tensor explained in Appendix B. The interaction parts of this action have been derived by Maldacena<sup>4)</sup> up to the third-order terms. In particular, we will focus on an interaction between two scalars and a graviton. This is because the correlation between a small wave number of the tensor mode and large wave numbers of the scalar modes will remain despite the tensor mode decays after the mode reenters the cosmic horizon. We find a leading term of the two scalars and a graviton interaction in the action coming from the matter part of the Lagrangian through  $X$  as

$$X|_{\text{3rd-order}} \supset a^{-2} \frac{P_X}{2} \gamma_{ab} \partial_a \varphi \partial_b \varphi , \quad (3.5)$$

therefore, the interaction part is given by

$$S_{\text{int}} \supset \int d^4x \, a g_{tss} \gamma_{ab} \partial_a \zeta \partial_b \zeta . \quad (3.6)$$

Here, we introduce a coupling constant  $g_{tss}$ . From the definition of  $\zeta, \gamma_{ab}$  and the slow-roll parameter,  $g_{tss} = \epsilon$ . For a general consideration, let us deal with  $g_{tss}$  as a free parameter. In this sense, constraining on this parameter may offer a probe of the nature of inflation and gravity in the early universe. The primordial bispectrum is then computed using in-in formalism in the next subsection.

### 3.2. Calculation of the initial bispectrum

In the same manner as discussed in Ref. 4), we calculate the primordial bispectrum generated from two scalars and a graviton in the lowest order of the slow-roll parameter:

$$\begin{aligned} \left\langle \xi^{(\pm 2)}(\mathbf{k}_1) \xi^{(0)}(\mathbf{k}_2) \xi^{(0)}(\mathbf{k}_3) \right\rangle &= (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \frac{4g_{tss} I(k_1, k_2, k_3) k_2 k_3}{\prod_i (2k_i^3)} \frac{H_*^4}{2c_{s*}^2 \epsilon_*^2 M_{\text{pl}}^4} \\ &\quad \times e_{ab}^{(\mp 2)}(\hat{\mathbf{k}}_1) \hat{k}_{2a} \hat{k}_{3b} , \end{aligned} \quad (3.7)$$

$$I(k_1, k_2, k_3) \equiv -k_t + \frac{k_1 k_2 + k_2 k_3 + k_3 k_1}{k_t} + \frac{k_1 k_2 k_3}{k_t^2} , \quad (3.8)$$

where  $k_t \equiv k_1 + k_2 + k_3$ , and  $*$  means that it is evaluated at the time of horizon crossing, i.e.,  $a_* H_* = k$ . Here, we keep the angular and polarization dependences,  $e_{ab}^{(\mp 2)}(\hat{\mathbf{k}}_1) \hat{k}_{2a} \hat{k}_{3b}$ , which have sometimes been omitted in the literature for simplicity.<sup>26), 18), 13)</sup> We show, however, that expanding this term with spin-weighted spherical harmonics enables us to formulate the rotational-invariant bispectrum in an explicit way. The statistically isotropic power spectra of  $\xi^{(0)}$  and  $\xi^{(\pm 2)}$  are respectively

given by

$$\begin{aligned}
\langle \xi^{(0)}(\mathbf{k}) \xi^{(0)*}(\mathbf{k}') \rangle &\equiv (2\pi)^3 P_S(k) \delta(\mathbf{k} - \mathbf{k}') , \\
\frac{k^3 P_S(k)}{2\pi^2} &= \frac{H_*^2}{8\pi^2 c_{s*} \epsilon_* M_{\text{pl}}^2} \equiv A_S , \\
\langle \xi^{(\lambda)}(\mathbf{k}) \xi^{(\lambda')*}(\mathbf{k}') \rangle &\equiv (2\pi)^3 \frac{P_T(k)}{2} \delta(\mathbf{k} - \mathbf{k}') \delta_{\lambda, \lambda'} \text{ (for } \lambda = \pm 2 \text{)} , \\
\frac{k^3 P_T(k)}{2\pi^2} &= \frac{H_*^2}{\pi^2 M_{\text{pl}}^2} = 8c_{s*} \epsilon_* A_S \equiv \frac{r}{2} A_S , \tag{3.9}
\end{aligned}$$

where  $r$  is the tensor-to-scalar ratio and  $A_S$  is the amplitude of primordial curvature perturbations. Note that the power spectra satisfy the scale invariance because we consider them in the lowest order of the slow-roll parameter. Using these equations, we parametrize the initial bispectrum in this case from Eqs. (3.7) and (2.5) as

$$F^{\pm 200}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = f^{(TSS)}(k_1, k_2, k_3) e_{ab}^{(\mp 2)}(\hat{\mathbf{k}}_1) \hat{k}_{2a} \hat{k}_{3b} , \tag{3.10}$$

$$f^{(TSS)}(k_1, k_2, k_3) \equiv \frac{16\pi^4 A_S^2 g_{tss}}{k_1^2 k_2^2 k_3^2} \frac{I(k_1, k_2, k_3)}{k_t} \frac{k_t}{k_1} . \tag{3.11}$$

Note that  $f^{(TSS)}$  seems not to depend on the tensor-to-scalar ratio. In Fig. 1, we show the shape of  $I/k_1$ . From this, we confirm that the initial bispectrum  $f^{(TSS)}$  (3.11) dominates in the squeezed limit as  $k_1 \ll k_2 \simeq k_3$  like the local-type bispectrum of scalar modes.

In the squeezed limit, the ratio of  $f^{(TSS)}$  to the scalar-scalar-scalar counterpart  $f^{(SSS)} = \frac{6}{5} f_{\text{NL}} P_S(k_1) P_S(k_2)$ , which has been considered frequently, reads

$$\frac{f^{(TSS)}}{f^{(SSS)}} = \frac{10g_{tss}}{3f_{\text{NL}}} \frac{I}{k_t} \frac{k_t k_2}{k_3^2} \rightarrow \frac{20g_{tss}}{3f_{\text{NL}}} \frac{I}{k_t} . \tag{3.12}$$

In the standard slow-roll inflation model, this ratio becomes  $\mathcal{O}(1)$  and does not depend on the tensor-to-scalar ratio because  $g_{tss}$  and  $f_{\text{NL}}$  are proportional to the slow-roll parameter  $\epsilon$ , and  $I/k_t$  has a nearly flat shape. The average of amplitude is evaluated as  $I/k_t \approx -0.6537$ . Therefore, it manifests the comparable importance of the higher order correlations of tensor modes to the scalar ones in the standard inflation scenario.

### 3.3. Formulation of the CMB bispectrum

For this case, by substituting Eq. (3.10) into Eq. (2.7), the initial bispectrum is given by

$$\begin{aligned}
\mathcal{F}_{\ell_1 \ell_2 \ell_3}^{\pm 200}(k_1, k_2, k_3) &= \sum_{m_1 m_2 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \left( \prod_{i=1}^3 \int d^2 \hat{\mathbf{k}}_i \right) \\
&\times {}_{\mp 2} Y_{\ell_1 m_1}^*(\hat{\mathbf{k}}_1) Y_{\ell_2 m_2}^*(\hat{\mathbf{k}}_2) Y_{\ell_3 m_3}^*(\hat{\mathbf{k}}_3) \\
&\times f^{(TSS)}(k_1, k_2, k_3) e_{ab}^{(\mp 2)}(\hat{\mathbf{k}}_1) \hat{k}_{2a} \hat{k}_{3b} \delta \left( \sum_{i=1}^3 \mathbf{k}_i \right) . \tag{3.13}
\end{aligned}$$

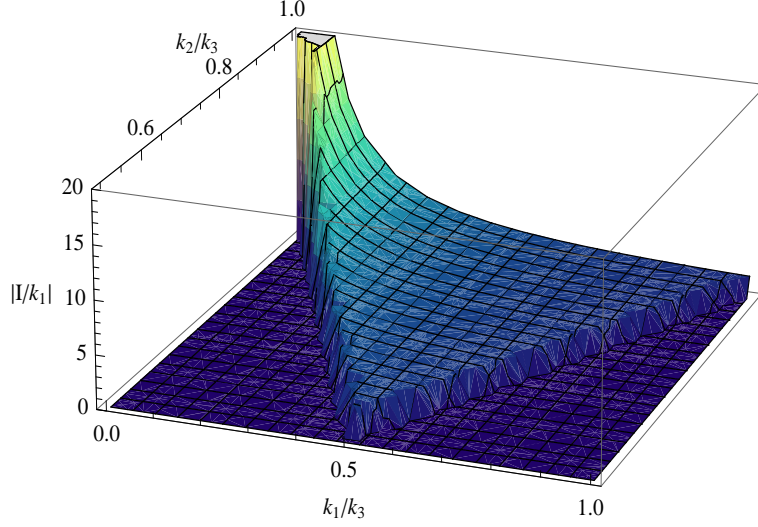


Fig. 1. (color online) Shape of  $I/k_1$ . For the symmetric property and the triangle condition, we limit the plot range as  $k_1 \leq k_2 \leq k_3$  and  $|k_1 - k_2| \leq k_3 \leq k_1 + k_2$ .

We derive this simpler form as the following procedure.

At first, we express all parts containing the angular dependence with the spin spherical harmonics:

$$e_{ab}^{(\mp 2)}(\hat{\mathbf{k}}_1)\hat{k}_{2a}\hat{k}_{3b} = \frac{4(8\pi)^{3/2}}{3} \sum_{M m_a m_b} \pm 2 Y_{2M}^*(\hat{\mathbf{k}}_1) Y_{1m_a}^*(\hat{\mathbf{k}}_2) Y_{1m_b}^*(\hat{\mathbf{k}}_3) \begin{pmatrix} 2 & 1 & 1 \\ M & m_a & m_b \end{pmatrix}, \quad (3.14)$$

$$\delta \left( \sum_{i=1}^3 \mathbf{k}_i \right) = 8 \int_0^\infty y^2 dy \left[ \prod_{i=1}^3 \sum_{L_i M_i} (-1)^{L_i/2} j_{L_i}(k_i y) Y_{L_i M_i}^*(\hat{\mathbf{k}}_i) \right] \\ \times I_{L_1 L_2 L_3}^{0 \ 0 \ 0} \begin{pmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \end{pmatrix}, \quad (3.15)$$

where we used the relations listed in Appendices A and B and

$$I_{l_1 l_2 l_3}^{s_1 s_2 s_3} \equiv \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ s_1 & s_2 & s_3 \end{pmatrix}. \quad (3.16)$$

Secondly, using Eq. (A.8), we replace all the integrals of spin spherical harmonics with the Wigner symbols:

$$\int d^2 \hat{\mathbf{k}}_1 \mp 2 Y_{\ell_1 m_1}^*(\hat{\mathbf{k}}_1) Y_{L_1 M_1}^*(\hat{\mathbf{k}}_1) \pm 2 Y_{2M}^*(\hat{\mathbf{k}}_1) = I_{\ell_1 L_1 2}^{\pm 20 \mp 2} \begin{pmatrix} \ell_1 & L_1 & 2 \\ m_1 & M_1 & M \end{pmatrix} \quad (3.17)$$

$$\int d^2 \hat{\mathbf{k}}_2 Y_{\ell_2 m_2}^*(\hat{\mathbf{k}}_2) Y_{L_2 M_2}^*(\hat{\mathbf{k}}_2) Y_{1m_a}^*(\hat{\mathbf{k}}_2) = I_{\ell_2 L_2 1}^{0 \ 0 \ 0} \begin{pmatrix} \ell_2 & L_2 & 1 \\ m_2 & M_2 & m_a \end{pmatrix}, \quad (3.18)$$

$$\int d^2 \hat{\mathbf{k}}_3 Y_{\ell_3 m_3}^*(\hat{\mathbf{k}}_3) Y_{L_3 M_3}^*(\hat{\mathbf{k}}_3) Y_{1m_b}^*(\hat{\mathbf{k}}_3) = I_{\ell_3 L_3 1}^{0 \ 0 \ 0} \begin{pmatrix} \ell_3 & L_3 & 1 \\ m_3 & M_3 & m_b \end{pmatrix}. \quad (3.19)$$

Thirdly, using the summation formula of five Wigner-3j symbols as Eq. (A·20), we sum up the Wigner-3j symbols with respect to azimuthal quantum numbers in the above equations and express with the Wigner-9j symbol as

$$\begin{aligned} & \sum_{\substack{M_1 M_2 M_3 \\ M m_a m_b}} \begin{pmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ M & m_a & m_b \end{pmatrix} \\ & \times \begin{pmatrix} \ell_1 & L_1 & 2 \\ m_1 & M_1 & M \end{pmatrix} \begin{pmatrix} \ell_2 & L_2 & 1 \\ m_2 & M_2 & m_a \end{pmatrix} \begin{pmatrix} \ell_3 & L_3 & 1 \\ m_3 & M_3 & m_b \end{pmatrix} \\ & = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \left\{ \begin{matrix} \ell_1 & \ell_2 & \ell_3 \\ L_1 & L_2 & L_3 \\ 2 & 1 & 1 \end{matrix} \right\}. \end{aligned} \quad (3\cdot20)$$

After these treatments, performing the summation over  $m_1, m_2$  and  $m_3$  like Eq. (A·5), we can obtain the final form as

$$\begin{aligned} \mathcal{F}_{\ell_1 \ell_2 \ell_3}^{\pm 200}(k_1, k_2, k_3) &= \frac{(8\pi)^{3/2}}{6} f^{(TSS)}(k_1, k_2, k_3) \\ &\times \sum_{L_1 L_2 L_3} I_{L_1 L_2 L_3}^{0\ 0\ 0} I_{\ell_1 L_1 2}^{\pm 20 \mp 2} I_{\ell_2 L_2 1}^{0\ 0\ 0} I_{\ell_3 L_3 1}^{0\ 0\ 0} \left\{ \begin{matrix} \ell_1 & \ell_2 & \ell_3 \\ L_1 & L_2 & L_3 \\ 2 & 1 & 1 \end{matrix} \right\} \\ &\times \int_0^\infty y^2 dy \left[ \prod_{i=1}^3 (-1)^{L_i/2} j_{L_i}(k_i y) \right]. \end{aligned} \quad (3\cdot21)$$

Note that the absence of the summation over  $m_1, m_2$  and  $m_3$  in this equation means that the tensor-scalar-scalar bispectrum maintains the rotational invariance. As described above, this consequence is derived from the angular dependence in the polarization tensor. Also in vector modes, if their power spectra obey the statistical isotropy like Eq. (3·9), one can obtain the rotational invariant bispectrum by considering the angular dependence in the polarization vector as Eq. (B·12).

Then, substituting the expression (3·21) into Eq. (2·9), we can calculate the CMB bispectrum induced from the nonlinear coupling between two scalars and a graviton. The CMB angle-averaged bispectrum is derived as

$$\begin{aligned} & B_{X_1 X_2 X_3, \ell_1 \ell_2 \ell_3}^{(TSS)} \\ &= \frac{(8\pi)^{3/2}}{3} \sum_{L_1 L_2 L_3} (-1)^{\frac{L_1+L_2+L_3}{2}} I_{L_1 L_2 L_3}^{0\ 0\ 0} I_{\ell_1 L_1 2}^{20-2} I_{\ell_2 L_2 1}^{0\ 0\ 0} I_{\ell_3 L_3 1}^{0\ 0\ 0} \left\{ \begin{matrix} \ell_1 & \ell_2 & \ell_3 \\ L_1 & L_2 & L_3 \\ 2 & 1 & 1 \end{matrix} \right\} \\ &\times \int_0^\infty y^2 dy \left[ \prod_{n=1}^3 \frac{2}{\pi} (-i)^{\ell_n} \int_0^\infty k_n^2 dk_n \mathcal{T}_{X_n, \ell_n}^{(Z_n)} j_{L_n}(k_n y) \right] f^{(TSS)}(k_1, k_2, k_3), \end{aligned} \quad (3\cdot22)$$

where we use the summation over  $\lambda_1 = \pm 2$  as

$$\sum_{\lambda_1 = \pm 2} [\text{sgn}(\lambda_1)]^{\lambda_1 + x_1} I_{\ell_1 L_1 2}^{\lambda_1 0 - \lambda_1} = \begin{cases} 2 I_{\ell_1 L_1 2}^{20-2} & (\text{for } x_1 + L_1 + \ell_1 = \text{even}) , \\ 0 & (\text{for } x_1 + L_1 + \ell_1 = \text{odd}) . \end{cases} \quad (3\cdot23)$$



Considering the selection rules of the Wigner symbols explained in Appendix A, we see that the bispectrum (3.22) has nonzero value under the conditions:

$$\begin{aligned}
 L_1 &= \begin{cases} |\ell_1 \pm 2|, \ell_1 & (\text{for } X_1 = I, E) \\ |\ell_1 \pm 1| & (\text{for } X_1 = B) \end{cases}, \quad L_2 = |\ell_2 \pm 1|, \quad L_3 = |\ell_3 \pm 1|, \\
 |L_1 - L_2| &\leq L_3 \leq L_1 + L_2, \quad \sum_{i=1}^3 L_i = \text{even}, \\
 |\ell_1 - \ell_2| &\leq \ell_3 \leq \ell_1 + \ell_2, \quad \sum_{i=1}^3 \ell_i = \begin{cases} \text{even} & (\text{for } X_1 = I, E) \\ \text{odd} & (\text{for } X_1 = B) \end{cases}. \quad (3.24)
 \end{aligned}$$

In Figs. 2 and 3, we describe the reduced CMB bispectra of intensity mode sourced from two scalars and a graviton coupling:

$$\begin{aligned}
 &b_{III, \ell_1 \ell_2 \ell_3}^{(TSS)} + b_{III, \ell_1 \ell_2 \ell_3}^{(STS)} + b_{III, \ell_1 \ell_2 \ell_3}^{(SST)} \\
 &= (I_{\ell_1 \ell_2 \ell_3}^0)^{-1} \left( B_{III, \ell_1 \ell_2 \ell_3}^{(TSS)} + B_{III, \ell_1 \ell_2 \ell_3}^{(STS)} + B_{III, \ell_1 \ell_2 \ell_3}^{(SST)} \right), \quad (3.25)
 \end{aligned}$$

and primordial curvature perturbations:

$$b_{III, \ell_1 \ell_2 \ell_3}^{(SSS)} = (I_{\ell_1 \ell_2 \ell_3}^0)^{-1} B_{III, \ell_1 \ell_2 \ell_3}^{(SSS)}. \quad (3.26)$$

For the numerical computation, we modify the Boltzmann Code for Anisotropies in the Microwave Background (CAMB).<sup>20),27)</sup> In the calculation of the Wigner-3j and 9j symbols, we use the Common Mathematical Library SLATEC<sup>28)</sup> and the summation formula of three Wigner-6j symbols (A.21). As the radiation transfer functions of scalar and tensor modes, namely,  $\mathcal{T}_{X_i, \ell_i}^{(S)}$  and  $\mathcal{T}_{X_i, \ell_i}^{(T)}$ , we use the 1st-order formulae as discussed in Refs. 22) and 27). From the behavior of each line shown in Fig. 3 at small  $\ell_3$  that the reduced CMB bispectrum is roughly proportional to  $\ell^{-2}$ , we can confirm that the tensor-scalar-scalar bispectrum has a nearly squeezed-type configuration corresponding to the shape of the initial bispectrum as discussed above. From Fig. 2, by comparing the green dashed line with the red solid line roughly estimated as

$$|b_{III, \ell \ell \ell}^{(TSS)} + b_{III, \ell \ell \ell}^{(STS)} + b_{III, \ell \ell \ell}^{(SST)}| \sim \ell^{-4} \times 8 \times 10^{-18} |g_{tss}|, \quad (3.27)$$

we find that  $|g_{tss}| \sim 5$  is comparable to  $f_{\text{NL}}^{\text{local}} = 5$  corresponding to the upper bound expected from the PLANCK experiment. In the next subsection, we check the validity of the above evaluation by computation of the signal-to-noise ratio assuming the zero-noise data.

### 3.4. Estimation of the signal-to-noise ratio

Here, we compute the signal-to-noise ratio by comparing the intensity bispectrum of Eq. (3.22) with the zero-noise (ideal) data and examine the bound on the absolute value of  $g_{tss}$ . The formulation of (the square of) the signal-to-noise ratio

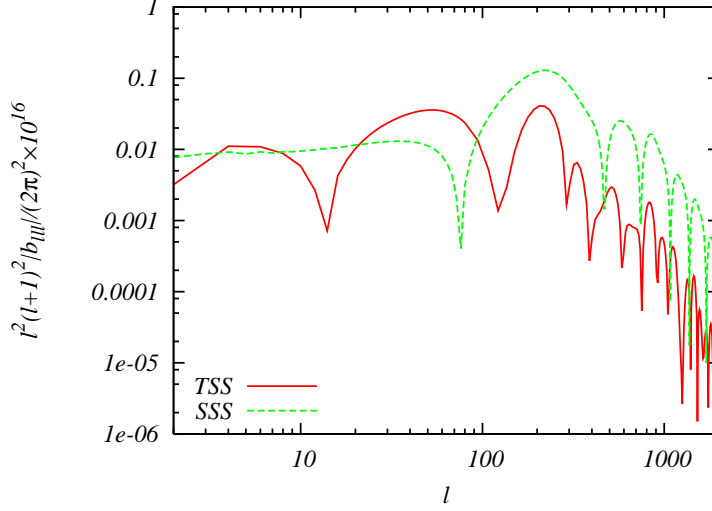


Fig. 2. (color online) Absolute values of the CMB reduced bispectra of temperature fluctuation for  $\ell_1 = \ell_2 = \ell_3$ . The lines correspond to the spectra generated from tensor-scalar-scalar correlation given by Eq. (3.25) with  $g_{tss} = 5$  (red solid line) and the primordial non-Gaussianity in the scalar curvature perturbations with  $f_{NL}^{\text{local}} = 5$  (green dashed line). The other cosmological parameters are fixed to the mean values limited from WMAP-7yr data reported in Ref. 3).

( $S/N$ ) is reported in Refs. 1) and 2). In our case, it can be expressed as

$$\left(\frac{S}{N}\right)^2 = \sum_{2 \leq \ell_1 \leq \ell_2 \leq \ell_3 \leq \ell} \frac{\left(B_{III\ell_1\ell_2\ell_3}^{(TSS)} + B_{III\ell_1\ell_2\ell_3}^{(STS)} + B_{III\ell_1\ell_2\ell_3}^{(SST)}\right)^2}{\sigma_{\ell_1\ell_2\ell_3}^2}, \quad (3.28)$$

where  $\sigma_{\ell_1\ell_2\ell_3}$  denotes the variance of the bispectrum. Assuming the weakly non-Gaussianity, the variance can be estimated as<sup>29),30)</sup>

$$\sigma_{\ell_1\ell_2\ell_3}^2 \approx C_{\ell_1} C_{\ell_2} C_{\ell_3} \Delta_{\ell_1\ell_2\ell_3}, \quad (3.29)$$

where  $\Delta_{\ell_1\ell_2\ell_3}$  takes 1, 6 or 2 for  $\ell_1 \neq \ell_2 \neq \ell_3, \ell_1 = \ell_2 = \ell_3$ , or the case that two  $\ell$ 's are the same, respectively.  $C_\ell$  denotes that the CMB angular power spectrum included the noise spectrum, which is neglected in our case.

In Fig. 4, the numerical result of Eq. (3.28) is presented. We find that ( $S/N$ ) is a monotonically increasing function roughly proportional to  $\ell$  for  $\ell < 2000$ . It is compared with the order estimation of Eq. (3.28) as Ref. 2)

$$\begin{aligned} \left(\frac{S}{N}\right) &\sim \sqrt{\frac{\ell^3}{24}} \times \sqrt{\frac{(2\ell)^3}{4\pi}} \left| \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix} \right| \frac{\ell^3 |b_{III\ell\ell\ell}^{(TSS)} + b_{III\ell\ell\ell}^{(STS)} + b_{III\ell\ell\ell}^{(SST)}|}{(\ell^2 C_\ell)^{3/2}} \\ &\sim \ell \times 5.4 \times 10^{-5} |g_{tss}|. \end{aligned} \quad (3.30)$$

Here, we use Eq. (3.27) and the approximations as  $\sum \sim \ell^3/24$ ,  $\ell^3 \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}^2 \sim 0.36 \times \ell$ , and  $\ell^2 C_\ell \sim 6 \times 10^{-10}$ . We confirm that this is consistent with Fig. 4,

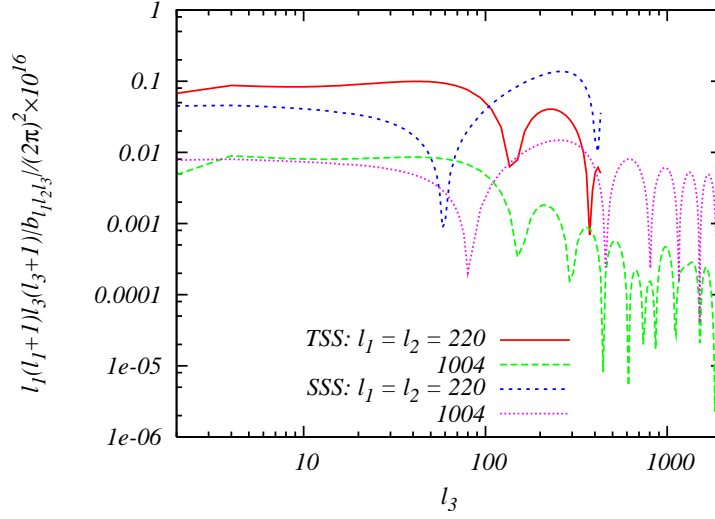


Fig. 3. (color online) Absolute values of the CMB reduced bispectra of temperature fluctuation generated from tensor-scalar-scalar correlation given by Eq. (3.25) (*TSS*) and the primordial non-Gaussianity in the scalar curvature perturbations (*SSS*) as a function of  $\ell_3$  with  $\ell_1$  and  $\ell_2$  fixed to some values as indicated. The parameters are fixed to the same values defined in Fig. 2.

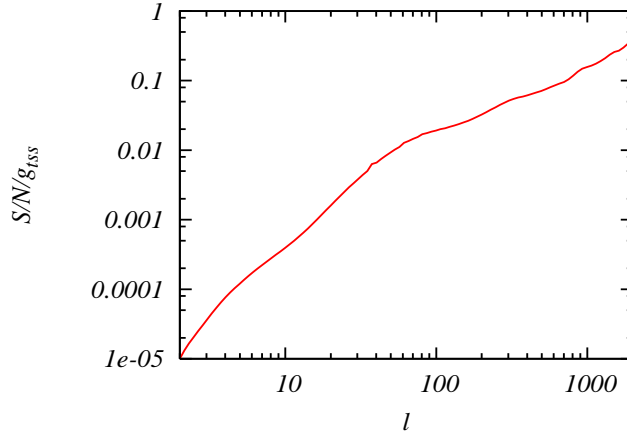


Fig. 4. (color online) Signal-to-noise ratio normalized by  $g_{tss}$  as a function of the maximum value between  $\ell_1, \ell_2$  and  $\ell_3$ , namely,  $\ell$ . Each parameter is fixed to the same values defined in Fig. 2.

which justifies our numerical calculation in some sense. This figure shows that from the WMAP and PLANCK experimental data,<sup>3),31)</sup> which are roughly noise-free at  $\ell \lesssim 500$  and  $1000$ , respectively, expected  $(S/N)/g_{tss}$  values are  $0.072$  and  $0.16$ . Hence, to obtain  $(S/N) > 1$ , we need  $|g_{tss}| > 14$  and  $6$ . The latter value is consistent with a naive estimate  $|g_{tss}| \lesssim 5$ , which was discussed at the end of the previous section.

#### §4. Summary and discussion

In this paper, we present a full-sky formalism of the CMB bispectrum sourced from the primordial non-Gaussianity not only in the scalar but also in the vector and tensor perturbations. As an extension of the previous formalism discussed in Ref. 18), the new formalism contains the contribution of the polarization vector and tensor in the initial bispectrum. In Ref. 18), we have shown that in the all-sky analysis, the CMB bispectrum of vector or tensor mode cannot be formed as a simple angle-averaged bispectrum in the same way as that of scalar mode. This is because the angular integrals over the wave number vectors have complexities for the non-orthogonality of spin spherical harmonics whose spin values differ from each other if one neglects the angular dependence of the polarization vector or tensor. In this study, however, we find that this difficulty vanishes if we maintain the angular dependence in the initial bispectrum.

To present how to use our formalism, we compute the CMB bispectrum induced by the nonlinear mode-coupling between the two scalars and a graviton.<sup>4)</sup> The typical value of the reduced bispectrum in temperature fluctuations is calculated as a function of the coupling constant between scalars and gravitons  $g_{tss}$ :  $|b_{III,\ell\ell\ell}^{(TSS)} + b_{III,\ell\ell\ell}^{(STS)} + b_{III,\ell\ell\ell}^{(SST)}| \sim \ell^{-4} \times 8 \times 10^{-18} |g_{tss}|$ . Through the computation of the signal-to-noise ratio, we expect a constraint as  $|g_{tss}| < 14$  from WMAP and  $|g_{tss}| < 6$  from PLANCK. Although we do not include the effect of the polarization modes in the estimation of  $g_{tss}$  in this study, they will provide more beneficial information of the nonlinear nature of the early universe.

Our formalism will be applicable to the other sources of vector or tensor non-Gaussianity, such as, the cosmic strings<sup>9),10)</sup> or the primordial magnetic fields.<sup>17)</sup> Actually, in the specific case of vector-vector-vector correlation, we have already presented the rotationally invariant bispectrum from primordial magnetic fields.<sup>16)</sup>

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## Appendix A

### — Useful Properties of the Wigner Symbols —

Here, we briefly review the useful properties of the Wigner- $3j$ ,  $6j$  and  $9j$  symbols. The following discussions are based on Refs. 32), 33), 34), 35), 36).

#### A.1. Wigner- $3j$ symbol

In quantum mechanics, considering the coupling of two angular momenta as

$$\mathbf{l}_3 = \mathbf{l}_1 + \mathbf{l}_2, \quad (\text{A}\cdot 1)$$

the scalar product of eigenstates between the right-handed term and the left-handed one, namely, a Clebsch-Gordan coefficient, is related to the Wigner- $3j$  symbol:

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \equiv \frac{(-1)^{l_1-l_2+m_3} \langle l_1 m_1 l_2 m_2 | (l_1 l_2) l_3 m_3 \rangle}{\sqrt{2l_3+1}}. \quad (\text{A}\cdot 2)$$

This symbol vanishes unless the selection rules are satisfied as follows:

$$\begin{aligned} |m_1| \leq l_1, \quad |m_2| \leq l_2, \quad |m_3| \leq l_3, \quad m_1 + m_2 = m_3, \\ |l_1 - l_2| \leq l_3 \leq l_1 + l_2 \text{ (the triangle condition)}, \quad l_1 + l_2 + l_3 \in \mathbb{Z}. \end{aligned} \quad (\text{A}\cdot 3)$$

Symmetries of the Wigner- $3j$  symbol are given by

$$\begin{aligned} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= (-1)^{\sum_{i=1}^3 l_i} \begin{pmatrix} l_2 & l_1 & l_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-1)^{\sum_{i=1}^3 l_i} \begin{pmatrix} l_1 & l_3 & l_2 \\ m_1 & m_3 & m_2 \end{pmatrix} \\ &\quad \text{(odd permutation of columns)} \\ &= \begin{pmatrix} l_2 & l_3 & l_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} l_3 & l_1 & l_2 \\ m_3 & m_1 & m_2 \end{pmatrix} \\ &\quad \text{(even permutation of columns)} \\ &= (-1)^{\sum_{i=1}^3 l_i} \begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \\ &\quad \text{(sign inversion of } m_1, m_2, m_3 \text{)}. \end{aligned} \quad (\text{A}\cdot 4)$$

The Wigner- $3j$  symbols satisfy the orthogonality as

$$\begin{aligned} (2l_3+1) \sum_{l_3 m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m'_1 & m'_2 & m_3 \end{pmatrix} &= \delta_{m_1, m'_1} \delta_{m_2, m'_2}, \\ (2l_3+1) \sum_{m_1 m_2} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l'_3 \\ m_1 & m_2 & m'_3 \end{pmatrix} &= \delta_{l_3, l'_3} \delta_{m_3, m'_3}. \end{aligned} \quad (\text{A}\cdot 5)$$

For a special case that  $\sum_{i=1}^3 l_i = \text{even}$  and  $m_1 = m_2 = m_3 = 0$ , there is an analytical expression as

$$\begin{aligned} &\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \\ &= (-1)^{\sum_{i=1}^3 l_i} \frac{\left(\sum_{i=1}^3 \frac{l_i}{2}\right)! \sqrt{(-l_1+l_2+l_3)!} \sqrt{(l_1-l_2+l_3)!} \sqrt{(l_1+l_2-l_3)!}}{\left(\frac{-l_1+l_2+l_3}{2}\right)! \left(\frac{l_1-l_2+l_3}{2}\right)! \left(\frac{l_1+l_2-l_3}{2}\right)! \sqrt{\left(\sum_{i=1}^3 l_i + 1\right)!}} \end{aligned} \quad (\text{A}\cdot 6)$$

This vanishes for  $\sum_{i=1}^3 l_i = \text{odd}$ . The Wigner-3j symbol is related to the spin-weighted spherical harmonics as

$$\prod_{i=1}^2 s_i Y_{l_i m_i}(\hat{\mathbf{n}}) = \sum_{l_3 m_3 s_3} s_3 Y_{l_3 m_3}^*(\hat{\mathbf{n}}) I_{l_1 l_2 l_3}^{-s_1 - s_2 - s_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \quad (\text{A}\cdot 7)$$

which leads to the “extended” Gaunt integral including spin dependence:

$$\int d^2 \hat{\mathbf{n}} s_1 Y_{l_1 m_1}(\hat{\mathbf{n}}) s_2 Y_{l_2 m_2}(\hat{\mathbf{n}}) s_3 Y_{l_3 m_3}(\hat{\mathbf{n}}) = I_{l_1 l_2 l_3}^{-s_1 - s_2 - s_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (\text{A}\cdot 8)$$

$$\text{Here } I_{l_1 l_2 l_3}^{s_1 s_2 s_3} \equiv \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ s_1 & s_2 & s_3 \end{pmatrix}.$$

### A.2. Wigner-6j symbol

Considering two other ways in the coupling of three angular momenta as

$$\mathbf{l}_5 = \mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_4 \quad (\text{A}\cdot 9)$$

$$= \mathbf{l}_3 + \mathbf{l}_4 \quad (\text{A}\cdot 10)$$

$$= \mathbf{l}_1 + \mathbf{l}_6, \quad (\text{A}\cdot 11)$$

the Wigner-6j symbol is defined using a Clebsch-Gordan coefficient between each eigenstate of  $\mathbf{l}_5$  corresponding to Eqs. (A·10) and (A·11) as

$$\left\{ \begin{matrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{matrix} \right\} \equiv \frac{(-1)^{l_1+l_2+l_4+l_5} \langle (l_1 l_2) l_3; l_4; l_5 m_5 | l_1; (l_2 l_4) l_6; l_5 m_5 \rangle}{\sqrt{(2l_3+1)(2l_6+1)}}. \quad (\text{A}\cdot 12)$$

This is expressed with the summation of three Wigner-3j symbols:

$$\begin{aligned} & \sum_{m_4 m_5 m_6} (-1)^{\sum_{i=4}^6 l_i - m_i} \begin{pmatrix} l_5 & l_1 & l_6 \\ m_5 & -m_1 & -m_6 \end{pmatrix} \\ & \times \begin{pmatrix} l_6 & l_2 & l_4 \\ m_6 & -m_2 & -m_4 \end{pmatrix} \begin{pmatrix} l_4 & l_3 & l_5 \\ m_4 & -m_3 & -m_5 \end{pmatrix} \\ & = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \left\{ \begin{matrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{matrix} \right\}; \end{aligned} \quad (\text{A}\cdot 13)$$

hence, the triangle conditions are given by

$$\begin{aligned} |l_1 - l_2| &\leq l_3 \leq l_1 + l_2, \quad |l_4 - l_5| \leq l_3 \leq l_4 + l_5, \\ |l_1 - l_5| &\leq l_6 \leq l_1 + l_5, \quad |l_4 - l_2| \leq l_6 \leq l_4 + l_2. \end{aligned} \quad (\text{A}\cdot 14)$$

The Wigner-6j symbol obeys 24 symmetries such as

$$\begin{aligned} \left\{ \begin{matrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{matrix} \right\} &= \left\{ \begin{matrix} l_2 & l_1 & l_3 \\ l_5 & l_4 & l_6 \end{matrix} \right\} = \left\{ \begin{matrix} l_2 & l_3 & l_1 \\ l_5 & l_6 & l_4 \end{matrix} \right\} \quad (\text{permutation of columns}) \\ &= \left\{ \begin{matrix} l_4 & l_5 & l_3 \\ l_1 & l_2 & l_6 \end{matrix} \right\} = \left\{ \begin{matrix} l_1 & l_5 & l_6 \\ l_4 & l_2 & l_3 \end{matrix} \right\} \\ &\quad (\text{exchange of two corresponding elements between rows}). \end{aligned} \quad (\text{A}\cdot 15)$$

Geometrically, the Wigner-6j symbol is expressed using the tetrahedron composed of four triangles obeying Eq. (A.14). It is known that the Wigner-6j symbol is suppressed by the square root of the volume of the tetrahedron at high multipoles.

### A.3. Wigner-9j symbol

Considering two other ways in the coupling of four angular momenta as

$$\mathbf{l}_9 = \mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_4 + \mathbf{l}_5 \quad (\text{A.16})$$

$$= \mathbf{l}_3 + \mathbf{l}_6 \quad (\text{A.17})$$

$$= \mathbf{l}_7 + \mathbf{l}_8, \quad (\text{A.18})$$

where  $\mathbf{l}_3 \equiv \mathbf{l}_1 + \mathbf{l}_2$ ,  $\mathbf{l}_6 \equiv \mathbf{l}_4 + \mathbf{l}_5$ ,  $\mathbf{l}_7 \equiv \mathbf{l}_1 + \mathbf{l}_4$ ,  $\mathbf{l}_8 \equiv \mathbf{l}_2 + \mathbf{l}_5$ , the Wigner 9j symbol expresses a Clebsch-Gordan coefficient between each eigenstate of  $\mathbf{l}_9$  corresponding to Eqs. (A.17) and (A.18) as

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{array} \right\} \equiv \frac{\langle (l_1 l_2) l_3; (l_4 l_5) l_6; l_9 m_9 | (l_1 l_4) l_7; (l_2 l_5) l_8; l_9 m_9 \rangle}{\sqrt{(2l_3 + 1)(2l_6 + 1)(2l_7 + 1)(2l_8 + 1)}}. \quad (\text{A.19})$$

This is expressed with the summation of five Wigner-3j symbols:

$$\begin{aligned} & \sum_{\substack{m_4 m_5 m_6 \\ m_7 m_8 m_9}} \left( \begin{array}{ccc} l_4 & l_5 & l_6 \\ m_4 & m_5 & m_6 \end{array} \right) \left( \begin{array}{ccc} l_7 & l_8 & l_9 \\ m_7 & m_8 & m_9 \end{array} \right) \\ & \times \left( \begin{array}{ccc} l_4 & l_7 & l_1 \\ m_4 & m_7 & m_1 \end{array} \right) \left( \begin{array}{ccc} l_5 & l_8 & l_2 \\ m_5 & m_8 & m_2 \end{array} \right) \left( \begin{array}{ccc} l_6 & l_9 & l_3 \\ m_6 & m_9 & m_3 \end{array} \right) \\ & = \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{array} \right\}, \end{aligned} \quad (\text{A.20})$$

and that of three Wigner-6j symbols:

$$\begin{aligned} \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{array} \right\} &= \sum_x (-1)^{2x} (2x + 1) \\ & \times \left\{ \begin{array}{ccc} l_1 & l_4 & l_7 \\ l_8 & l_9 & x \end{array} \right\} \left\{ \begin{array}{ccc} l_2 & l_5 & l_8 \\ l_4 & x & l_6 \end{array} \right\} \left\{ \begin{array}{ccc} l_3 & l_6 & l_9 \\ x & l_1 & l_2 \end{array} \right\} \end{aligned} \quad (\text{A.21})$$

hence, the triangle conditions are given by

$$\begin{aligned} & |l_1 - l_2| \leq l_3 \leq l_1 + l_2, \quad |l_4 - l_5| \leq l_6 \leq l_4 + l_5, \quad |l_7 - l_8| \leq l_9 \leq l_7 + l_8, \\ & |l_1 - l_4| \leq l_7 \leq l_1 + l_4, \quad |l_2 - l_5| \leq l_8 \leq l_2 + l_5, \quad |l_3 - l_6| \leq l_9 \leq l_3 + l_6. \end{aligned} \quad (\text{A.22})$$

The Wigner-9j symbol obeys 72 symmetries:

$$\left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{array} \right\} = (-1)^{\sum_{i=1}^9 l_i} \left\{ \begin{array}{ccc} l_2 & l_1 & l_3 \\ l_5 & l_4 & l_6 \\ l_8 & l_7 & l_9 \end{array} \right\} = (-1)^{\sum_{i=1}^9 l_i} \left\{ \begin{array}{ccc} l_1 & l_2 & l_3 \\ l_7 & l_8 & l_9 \\ l_4 & l_5 & l_6 \end{array} \right\}$$

$$\begin{aligned}
& \text{(odd permutation of rows or columns)} \\
& = \begin{Bmatrix} l_2 & l_3 & l_1 \\ l_5 & l_6 & l_4 \\ l_8 & l_9 & l_7 \end{Bmatrix} = \begin{Bmatrix} l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \\ l_1 & l_2 & l_3 \end{Bmatrix} \\
& \text{(even permutation of rows or columns)} \\
& = \begin{Bmatrix} l_1 & l_4 & l_7 \\ l_2 & l_5 & l_8 \\ l_3 & l_6 & l_9 \end{Bmatrix} = \begin{Bmatrix} l_9 & l_6 & l_3 \\ l_8 & l_5 & l_2 \\ l_7 & l_4 & l_1 \end{Bmatrix} \\
& \text{(reflection of the symbols)} . \tag{A.23}
\end{aligned}$$

## Appendix B

### —— Polarization Vector and Tensor ——

We summarize the relations and properties of a divergenceless polarization vector  $\epsilon_a^{(\pm 1)}$  and a transverse and traceless polarization tensor  $e_{ab}^{(\pm 2)37}$ .

The polarization vector with respect to a unit vector  $\hat{\mathbf{n}}$  is expressed using two unit vectors  $\hat{\boldsymbol{\theta}}$  and  $\hat{\boldsymbol{\phi}}$  perpendicular to  $\hat{\mathbf{n}}$  as

$$\epsilon_a^{(\pm 1)}(\hat{\mathbf{n}}) = \frac{1}{\sqrt{2}}[\hat{\theta}_a(\hat{\mathbf{n}}) \pm i \hat{\phi}_a(\hat{\mathbf{n}})] . \tag{B.1}$$

This satisfies the relations:

$$\begin{aligned}
& \hat{n}^a \epsilon_a^{(\pm 1)}(\hat{\mathbf{n}}) = 0 , \\
& \epsilon_a^{(\pm 1)*}(\hat{\mathbf{n}}) = \epsilon_a^{(\mp 1)}(\hat{\mathbf{n}}) = \epsilon_a^{(\pm 1)}(-\hat{\mathbf{n}}) , \\
& \epsilon_a^{(\lambda)}(\hat{\mathbf{n}}) \epsilon_a^{(\lambda')}(\hat{\mathbf{n}}) = \delta_{\lambda, -\lambda'} \quad (\text{for } \lambda, \lambda' = \pm 1) . \tag{B.2}
\end{aligned}$$

By defining a rotational matrix, which transforms a unit vector parallel to the  $z$ -axis, namely  $\hat{\mathbf{z}}$ , to  $\hat{\mathbf{n}}$ , as

$$S(\hat{\mathbf{n}}) \equiv \begin{pmatrix} \cos \theta_n \cos \phi_n & -\sin \phi_n & \sin \theta_n \cos \phi_n \\ \cos \theta_n \sin \phi_n & \cos \phi_n & \sin \theta_n \sin \phi_n \\ -\sin \theta_n & 0 & \cos \theta_n \end{pmatrix} , \tag{B.3}$$

we specify  $\hat{\boldsymbol{\theta}}$  and  $\hat{\boldsymbol{\phi}}$  as

$$\hat{\boldsymbol{\theta}}(\hat{\mathbf{n}}) = S(\hat{\mathbf{n}})\hat{\mathbf{x}} , \quad \hat{\boldsymbol{\phi}}(\hat{\mathbf{n}}) = S(\hat{\mathbf{n}})\hat{\mathbf{y}} , \tag{B.4}$$

where  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}$  are unit vectors parallel to  $x$ - and  $y$ -axes. By using Eq. (B.1), the polarization tensor is constructed as

$$e_{ab}^{(\pm 2)}(\hat{\mathbf{n}}) = \sqrt{2} \epsilon_a^{(\pm 1)}(\hat{\mathbf{n}}) \epsilon_b^{(\pm 1)}(\hat{\mathbf{n}}) . \tag{B.5}$$

To utilize the polarization vector and tensor in the calculation of this paper, we need to expand Eqs. (B.1) and (B.5) with spin spherical harmonics. An arbitrary



unit vector is expanded with the spin-0 spherical harmonics as

$$\begin{aligned}\hat{r}_a &= \sum_m \alpha_a^m Y_{1m}(\hat{\mathbf{r}}) , \\ \alpha_a^m &\equiv \sqrt{\frac{2\pi}{3}} \begin{pmatrix} -m(\delta_{m,1} + \delta_{m,-1}) \\ i(\delta_{m,1} + \delta_{m,-1}) \\ \sqrt{2}\delta_{m,0} \end{pmatrix} .\end{aligned}\quad (\text{B}\cdot 6)$$

Here, note that the repeat of the index implies the summation. The scalar product of  $\alpha_a^m$  is calculated as

$$\alpha_a^m \alpha_a^{m'} = \frac{4\pi}{3} (-1)^m \delta_{m,-m'} , \quad \alpha_a^m \alpha_a^{m'*} = \frac{4\pi}{3} \delta_{m,m'} . \quad (\text{B}\cdot 7)$$

Through the substitution of Eq. (B.4) into Eq. (B.6),  $\hat{\boldsymbol{\theta}}$  is expanded as

$$\begin{aligned}\hat{\theta}_a(\hat{\mathbf{n}}) &= \sum_m \alpha_a^m Y_{1m}(\hat{\boldsymbol{\theta}}(\hat{\mathbf{n}})) = \sum_m \alpha_a^m \sum_{m'} D_{mm'}^{(1)*}(S(\hat{\mathbf{n}})) Y_{1m'}(\hat{\mathbf{x}}) \\ &= -\frac{s}{\sqrt{2}} (\delta_{s,1} + \delta_{s,-1}) \sum_m \alpha_a^m s Y_{1m}(\hat{\mathbf{n}}) .\end{aligned}\quad (\text{B}\cdot 8)$$

Here, we use the properties of the Wigner  $D$ -matrix as<sup>18), 24), 32), 37)</sup>

$$Y_{\ell m}(S(\hat{\mathbf{n}})\hat{\mathbf{x}}) = \sum_{m'} D_{mm'}^{(\ell)*}(S(\hat{\mathbf{n}})) Y_{\ell m'}(\hat{\mathbf{x}}) , \quad (\text{B}\cdot 9)$$

$$D_{ms}^{(\ell)}(S(\hat{\mathbf{n}})) = \left[ \frac{4\pi}{2\ell+1} \right]^{1/2} (-1)^s {}_s Y_{\ell m}^*(\hat{\mathbf{n}}) . \quad (\text{B}\cdot 10)$$

In the same manner,  $\hat{\boldsymbol{\phi}}$  is also calculated as

$$\hat{\phi}_a(\hat{\mathbf{n}}) = \frac{i}{\sqrt{2}} (\delta_{s,1} + \delta_{s,-1}) \sum_m \alpha_a^m s Y_{1m}(\hat{\mathbf{n}}) ; \quad (\text{B}\cdot 11)$$

hence, the explicit form of Eq. (B.1) is calculated as

$$\epsilon_a^{(\pm 1)}(\hat{\mathbf{n}}) = \mp \sum_m \alpha_a^{m \pm 1} Y_{1m}(\hat{\mathbf{n}}) . \quad (\text{B}\cdot 12)$$

Substituting this into Eq. (B.5) and using the relations of Appendix A and  $I_{2\ 1\ 1}^{\mp 2 \pm 1 \pm 1} = \frac{3}{2\sqrt{\pi}}$ , the polarization tensor can also be expressed as

$$e_{ab}^{(\pm 2)}(\hat{\mathbf{n}}) = \frac{3}{\sqrt{2\pi}} \sum_{M m_a m_b} \mp 2 Y_{2M}^*(\hat{\mathbf{n}}) \alpha_a^{m_a} \alpha_b^{m_b} \begin{pmatrix} 2 & 1 & 1 \\ M & m_a & m_b \end{pmatrix} . \quad (\text{B}\cdot 13)$$

This obeys the relations:

$$\begin{aligned}e_{aa}^{(\pm 2)}(\hat{\mathbf{n}}) &= \hat{n}_a e_{ab}^{(\pm 2)}(\hat{\mathbf{n}}) = 0 , \\ e_{ab}^{(\pm 2)*}(\hat{\mathbf{n}}) &= e_{ab}^{(\mp 2)}(\hat{\mathbf{n}}) = e_{ab}^{(\pm 2)}(-\hat{\mathbf{n}}) , \\ e_{ab}^{(\lambda)}(\hat{\mathbf{n}}) e_{ab}^{(\lambda')}(\hat{\mathbf{n}}) &= 2\delta_{\lambda, -\lambda'} \quad (\text{for } \lambda, \lambda' = \pm 2) .\end{aligned}\quad (\text{B}\cdot 14)$$

Using the projection operators as

$$O_a^{(0)} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv k^{-1} \nabla_a e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{k}_a e^{i\mathbf{k}\cdot\mathbf{x}} , \quad (\text{B}\cdot 15)$$

$$O_{ab}^{(0)} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv \left( k^{-2} \nabla_a \nabla_b + \frac{\delta_{a,b}}{3} \right) e^{i\mathbf{k}\cdot\mathbf{x}} = \left( -\hat{k}_a \hat{k}_b + \frac{\delta_{a,b}}{3} \right) e^{i\mathbf{k}\cdot\mathbf{x}} , \quad (\text{B}\cdot 16)$$

$$O_a^{(\pm 1)} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv -i\epsilon_a^{(\pm 1)}(\hat{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}} , \quad (\text{B}\cdot 17)$$

$$O_{ab}^{(\pm 1)} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv k^{-1} \nabla_a O_b^{(\pm 1)} e^{i\mathbf{k}\cdot\mathbf{x}} = \hat{k}_a \epsilon_b^{(\pm 1)}(\hat{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}} , \quad (\text{B}\cdot 18)$$

$$O_{ab}^{(\pm 2)} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv e_{ab}^{(\pm 2)}(\hat{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{x}} , \quad (\text{B}\cdot 19)$$

the arbitrary scalar, vector and tensor are decomposed into the helicity states as

$$\eta(\mathbf{k}) = \eta^{(0)}(\mathbf{k}) , \quad (\text{B}\cdot 20)$$

$$\omega_a(\mathbf{k}) = \omega^{(0)}(\mathbf{k}) O_a^{(0)} + \sum_{\lambda=\pm 1} \omega^{(\lambda)}(\mathbf{k}) O_a^{(\lambda)} , \quad (\text{B}\cdot 21)$$

$$\chi_{ab}(\mathbf{k}) = \chi^{(0)}(\mathbf{k}) O_{ab}^{(0)} + \sum_{\lambda=\pm 1} \chi^{(\lambda)}(\mathbf{k}) O_{ab}^{(\lambda)} + \sum_{\lambda=\pm 2} \chi^{(\lambda)}(\mathbf{k}) O_{ab}^{(\lambda)} . \quad (\text{B}\cdot 22)$$

Then, using Eq. (B·2) and (B·14), we can find the inverse formulae as

$$\omega^{(0)}(\mathbf{k}) = -i\hat{k}_a \omega_a(\mathbf{k}) , \quad (\text{B}\cdot 23)$$

$$\omega^{(\pm 1)}(\mathbf{k}) = i\epsilon_a^{(\mp 1)}(\hat{\mathbf{k}}) \omega_a(\mathbf{k}) , \quad (\text{B}\cdot 24)$$

$$\chi^{(0)}(\mathbf{k}) = \frac{3}{2} \left( -\hat{k}_a \hat{k}_b + \frac{\delta_{a,b}}{3} \right) \chi_{ab}(\mathbf{k}) , \quad (\text{B}\cdot 25)$$

$$\chi^{(\pm 1)}(\mathbf{k}) = \hat{k}_a \epsilon_b^{(\mp 1)}(\hat{\mathbf{k}}) \chi_{ab}(\mathbf{k}) , \quad (\text{B}\cdot 26)$$

$$\chi^{(\pm 2)}(\mathbf{k}) = \frac{1}{2} e_{ab}^{(\mp 2)}(\hat{\mathbf{k}}) \chi_{ab}(\mathbf{k}) . \quad (\text{B}\cdot 27)$$

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